

Liouville theorems, universal estimates and periodic solutions for cooperative parabolic Lotka-Volterra systems

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Abstract

We consider positive solutions of cooperative parabolic Lotka-Volterra systems with equal diffusion coefficients, in bounded and unbounded domains. The systems are complemented by the Dirichlet or Neumann boundary conditions. Under suitable assumptions on the coefficients of the reaction terms, these problems possess both global solutions and solutions which blow up in finite time. We show that any solution (u, v) defined on the time interval $(0, T)$ satisfies a universal estimate of the form

$$u(x, t) + v(x, t) \leq C(1 + t^{-1} + (T - t)^{-1}),$$

where C does not depend on x, t, u, v, T . In particular, this bound guarantees global existence and boundedness for threshold solutions lying on the borderline between blow-up and global existence. Moreover, this bound yields optimal blow-up rate estimates for solutions which blow up in finite time. Our estimates are based on new Liouville-type theorems for the corresponding scaling invariant parabolic system and require an optimal restriction on the space dimension n : $n \leq 5$. As an application we also prove the existence of time-periodic positive solutions if the coefficients are time-periodic. Our approach can also be used for more general parabolic systems.

1 Introduction

We consider nonnegative solutions of the Lotka-Volterra system

$$\left. \begin{aligned} u_t - d_1 \Delta u &= u(a_1 - b_1 u + c_1 v), \\ v_t - d_2 \Delta v &= v(a_2 - b_2 v + c_2 u), \end{aligned} \right\} \quad x \in \Omega, \quad t \in (0, T), \quad (1)$$

where Ω is a (possibly unbounded) domain in \mathbb{R}^n with a uniformly C^2 smooth boundary $\partial\Omega$, $T \in (0, \infty]$, d_1, d_2 are positive constants and

$$a_i, b_i, c_i \in L^\infty(\Omega \times (0, \infty)) \quad \text{for } i = 1, 2. \quad (2)$$

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Except for some marginal results in Theorem 7 and Remark 8 we will always assume

$$b_1, b_2, c_1, c_2 > 0, \quad c_1 c_2 > b_1 b_2. \quad (3)$$

By ν we denote the outer unit normal on $\partial\Omega$ and by $(1)_D$ or $(1)_N$ we denote system (1) complemented by the Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega \times (0, T)$ or the Neumann boundary conditions $u_\nu = v_\nu = 0$ on $\partial\Omega \times (0, T)$, respectively. Notice that $(1)_D = (1)_N = (1)$ if $\Omega = \mathbb{R}^n$. If Ω is bounded then by Λ_1 we denote the least eigenvalue of the negative Dirichlet Laplacian in Ω . Except for Proposition 10, by a solution we will always mean a nonnegative classical solution.

First consider the Dirichlet problem $(1)_D$ and assume that Ω is bounded and the coefficients a_i, b_i, c_i are constant. Then some solutions of $(1)_D$ blow up in finite time, see [13, the proof of Theorem 12.6.1]. Blow-up rates of such solutions have been studied in [8, 7] but an upper estimate of the rate (which is usually much more difficult than a lower estimate) has only been established if $n = 1$. Under suitable additional assumptions, $(1)_D$ possesses also nontrivial global solutions and steady states: If we assume $a_1/d_1, a_2/d_2 < \Lambda_1$, for example, then the existence of global solutions follows from the stability of the zero solution. If, in addition, $n \leq 5$ then there exists a positive steady state, and the assumption $n \leq 5$ is also necessary if Ω is starshaped and $a_1/d_1 = a_2/d_2$, see [10].

If one considers the Neumann problem $(1)_N$ with Ω bounded and a_i, b_i, c_i constant then some solutions blow up again and even “diffusion-induced blow-up” occurs: There exist blow-up solutions of $(1)_N$ such that the solutions of the corresponding system of ODEs exist globally, see [11]. On the other hand, nontrivial global solutions also exist if $a_1, a_2 < 0$, for example.

The existence of blow-up and global solutions of system (1) is also known in the case of non-constant coefficients, see [9] and the references therein, for example. If the problem $(1)_D$ or $(1)_N$ possesses both global solutions and solutions which blow up in finite time then one can study so-called threshold solutions, i.e. solutions lying on the borderline between global existence and blow-up. The study of such solutions is difficult even for the scalar problem

$$\left. \begin{aligned} u_t - \Delta u &= cu^2, & x \in \Omega, \quad t > 0, \\ u &= 0, & x \in \partial\Omega, \quad t > 0, \end{aligned} \right\} \quad (4)$$

where c is a positive constant and Ω is bounded. Notice that problem (4) is just a very special case of $(1)_D$ (if we set $u = v$, $d_1 = d_2 = 1$, $a_1 = a_2 = 0$, $b_1 = b_2 = 1$ and $c_1 = c_2 = c + 1$). All threshold solutions of (4) are global and bounded if $n \leq 5$ but they may be global and unbounded if $n = 6$ and they even may blow up in finite time if $n > 6$, see [18] and the references therein. It should be mentioned that the threshold solutions of (4) are — in some sense — the most interesting ones. In particular, any positive steady state of (4) is a threshold solution and the ω -limit set of any global bounded positive threshold solution of (4) is nonempty and consists of positive steady states. As far as we know, the behavior of threshold solutions of problems $(1)_D$ or $(1)_N$ has not been studied yet.

If $n \leq 5$ and the coefficient c in (4) is a positive periodic function of t then one can use a priori estimates of global solutions of (4) in order to prove the existence of nontrivial

periodic solutions, see [5, 16, 2]. Again, the existence of periodic solutions does not seem to be known for problems $(1)_D$ or $(1)_N$ with periodic coefficients in the presence of blow-up (see [9], for example, for the existence of periodic solutions of $(1)_D$ in the case $c_1 c_2 < b_1 b_2$ which excludes blow-up).

In this paper we will assume that $n \leq 5$, $d_1 = d_2 = 1$, the coefficients a_i, b_i, c_i satisfy (3) and suitable regularity assumptions and we will prove that all solutions of $(1)_D$ or $(1)_N$ satisfy universal a priori estimates of the form

$$u(x, t) + v(x, t) \leq C(1 + t^{-1} + (T - t)^{-1}), \quad x \in \Omega, \quad t \in (0, T), \quad (5)$$

where C does not depend on x, t, u, v, T . These estimates guarantee global existence and boundedness of threshold solutions and also optimal blow-up rate estimates of solutions which blow up in finite time. In addition, if Ω is bounded, a_i, b_i, c_i are time-periodic and $a_1, a_2 < \Lambda_1$ then these estimates combined with a homotopy argument guarantee the existence of a time-periodic positive solution of $(1)_D$. The homotopy used in our proof is quite different from those used for the scalar problem (4) in [5] or the steady-state problem for $(1)_D$ in [10].

Our estimates are based on Liouville-type theorems for corresponding scaling invariant systems. In fact, we will prove Liouville-type theorems for more general systems of the form (12). Their proofs rely on the fact that the components of any entire solution of such system have to be proportional, i.e. the problem can be reduced to a scalar problem. Arguments of this type have recently been used in [12, 19] in the case of elliptic systems.

2 Main results

We will first specify a class of coefficients of system (1) such that the constant C in the universal estimate (5) will not depend on the coefficients in this class. Fix $\varepsilon_0, M_0 > 0$ and a continuous function $\omega_0 : [0, \infty) \rightarrow [0, \infty)$ satisfying $\omega_0(0) = 0$, and set

$$\begin{aligned} \mathcal{F} = \mathcal{F}(\varepsilon_0, M_0, \omega_0) := \{ \phi : \Omega \times (0, \infty) \rightarrow [\varepsilon_0, M_0] : \\ |\phi(x, t) - \phi(y, s)| \leq \omega_0(|x - y| + |t - s|) \\ \text{for all } (x, t), (y, s) \in \Omega \times (0, \infty) \}. \end{aligned}$$

We assume that

$$\left. \begin{aligned} a_1, a_2 \in L^\infty(\Omega \times (0, \infty)) \quad \text{satisfy} \quad |a_1|, |a_2| \leq M_0, \\ b_1, b_2, c_1, c_2 \in \mathcal{F}(\varepsilon_0, M_0, \omega_0) \quad \text{satisfy} \quad c_1 c_2 \geq b_1 b_2 + \varepsilon_0, \end{aligned} \right\} \quad (6)$$

and consider system (1) with $d_1 = d_2 = 1$, i.e.

$$\left. \begin{aligned} u_t - \Delta u &= u(a_1 - b_1 u + c_1 v), \\ v_t - \Delta v &= v(a_2 - b_2 v + c_2 u), \end{aligned} \right\} \quad x \in \Omega, \quad t \in (0, T). \quad (7)$$

Our first result guarantees universal estimates of positive solutions of (7).

Theorem 1. *Let Ω be an arbitrary nonempty open subset of \mathbb{R}^n , $n \leq 5$, $T \in (0, \infty]$, $\varepsilon_0, M_0 > 0$, and $\omega_0 : [0, \infty) \rightarrow [0, \infty)$ be a continuous function satisfying $\omega_0(0) = 0$. Assume also that $a_1, a_2, b_1, b_2, c_1, c_2$ satisfy (6). Then there exists a positive constant $C = C(\varepsilon_0, M_0, \omega_0)$ such that any positive solution (u, v) of (7) satisfies*

$$u(x, t) + v(x, t) \leq C(C_1 + t^{-1} + (T - t)^{-1} + C_2 \text{dist}^{-2}(x, \partial\Omega)) \quad (8)$$

for all $(x, t) \in \Omega \times (0, T)$, where $C_1 = C_2 = 1$, the constant C is independent of $x, t, u, v, \Omega, T, a_1, a_2, b_1, b_2, c_1, c_2$,

$$(T - t)^{-1} := 0 \text{ if } T = \infty, \quad \text{dist}^{-2}(x, \partial\Omega) := 0 \text{ if } \Omega = \mathbb{R}^n,$$

and the solution (u, v) need not satisfy any boundary or initial condition.

If Ω is uniformly C^2 smooth and the solution (u, v) satisfies the homogeneous Dirichlet or Neumann boundary condition on $\partial\Omega \times (0, T)$ then (8) is true with $C = C(\Omega, \varepsilon_0, M_0, \omega_0)$, $C_1 = 1$ and $C_2 = 0$, i.e. estimate (5) is true.

If $a_1 = a_2 = 0$ and b_1, b_2, c_1, c_2 are constants, then (8) is true with $C_1 = 0$ and $C_2 = 1$.

In particular, Theorem 1 yields optimal upper blow-up rate estimates of solutions which blow up at time T , and also guarantees global existence and boundedness of threshold solutions.

In order to prove the existence of periodic solutions we fix $T \in (0, \infty)$ and assume that

$$\left. \begin{aligned} \Omega \subset \mathbb{R}^n \text{ is a } C^3\text{-smooth bounded domain, } n \leq 5, \\ a_i, b_i, c_i \in C(\overline{\Omega} \times [0, \infty)) \text{ are } T\text{-periodic in } t \text{ for } i = 1, 2, \\ b_1, b_2, c_1, c_2 > 0, \quad c_1 c_2 > b_1 b_2, \quad a_1, a_2 < \Lambda_1. \end{aligned} \right\} \quad (9)$$

Theorem 2. *Assume (9). Then system (7) complemented by the homogeneous Dirichlet boundary conditions possesses a positive T -periodic solution.*

Similarly as in [15], our estimates are based on doubling arguments, scaling and Liouville-type theorems for corresponding scaling invariant problems. In the case of system (7) the scaling invariant problem is

$$\left. \begin{aligned} u_t - \Delta u &= u(-b_1 u + c_1 v) \\ v_t - \Delta v &= v(-b_2 v + c_2 u) \end{aligned} \right\} \quad \text{in } X \times \mathbb{R}, \quad (10)$$

where

$$\left. \begin{aligned} \text{either } X &= \mathbb{R}^n \\ \text{or } X &= \mathbb{R}_+^n := \{x = (x_1, x_2, \dots, x_n) : x_1 > 0\}, \end{aligned} \right\} \quad (11)$$

b_1, b_2, c_1, c_2 are constants satisfying (3), and system (10) is complemented by the homogeneous Dirichlet or Neumann boundary conditions if $X = \mathbb{R}_+^n$. In fact, we will assume (11) and consider more general systems of the form

$$\left. \begin{aligned} u_t - \Delta u &= u^r(-b_1 u^q + c_1 v^q) \\ v_t - \Delta v &= v^r(-b_2 v^q + c_2 u^q), \end{aligned} \right\} \quad \text{in } X \times \mathbb{R}, \quad (12)$$

complemented by the Dirichlet or Neumann boundary conditions if $X = \mathbb{R}_+^n$. Here b_1, b_2, c_1, c_2 are constants satisfying (3), $q \geq r > 0$, $q + r > 1$ and

$$n \leq 2 \quad \text{or} \quad q + r < \frac{n(n+2)}{(n-1)^2}. \quad (13)$$

Theorem 3. *Assume (11). Let b_1, b_2, c_1, c_2 be constants satisfying (3) and q, r satisfy $q \geq r > 0$, $q + r > 1$ and (13). Let (u, v) be a nonnegative solution of (12) complemented by the homogeneous Dirichlet or Neumann boundary conditions if $X = \mathbb{R}_+^n$. In the case of the Dirichlet boundary condition assume also that (u, v) is bounded. Then $u \equiv v \equiv 0$.*

In the proof of Theorem 3 we first show that there exists $K > 0$ such that any nonnegative (or nonnegative bounded) solution satisfies $u = Kv$. This implies that u is a solution of the scalar equation

$$u_t - \Delta u = cu^{q+r} \quad \text{in } X \times \mathbb{R} \quad (14)$$

with some $c > 0$ (and u satisfies the corresponding boundary condition if $X = \mathbb{R}_+^n$). Now the Liouville theorems in [3, 15, 17] guarantee $u \equiv 0$ if (13) is true (or $q+r < (n+2)/(n-2)$ if we consider radial solutions only). Let us also mention that (14) possesses positive radial solutions if $X = \mathbb{R}^n$ and $q + r \geq (n+2)/(n-2)$.

If $r = q = 1$ (which corresponds to the Lotka-Volterra system (10)) then (13) can be written in the form $n \leq 5$ and condition $q + r \geq (n+2)/(n-2)$ is equivalent to $n \geq 6$, hence our Liouville theorems are optimal in this case.

Similarly, if $r = 1$, $q = 2$ then we obtain the nonexistence for (12) under the optimal condition $n \leq 3$ and — in the same way as in the case of $r = q = 1$ — one can use this result to obtain universal estimates for solutions of systems of the form

$$\left. \begin{aligned} u_t - \Delta u &= a_1 u - b_1 u^3 + c_1 uv^2, \\ v_t - \Delta v &= a_2 v - b_2 v^3 + c_2 u^2 v, \end{aligned} \right\} \quad x \in \Omega, \quad t \in (0, T), \quad (15)$$

complemented by the Dirichlet or Neumann boundary conditions. Notice that steady states of (15) correspond to the standing wave solutions of related Schrödinger systems, and estimates of global solutions of (15) can be useful in the study of the steady states, see [20]. Let us also mention that the nonexistence of positive solutions of (12) with $r = 1$, $q = 2$, $b_1 = b_2 = -1$ and $c_1 = c_2 > -1$ has recently been proved in [17] for $n \leq 2$ (or $n \leq 3$ in the class of radially symmetric solutions): The proof heavily used the gradient structure of the system. That result indicates that condition (3) is not necessary for the validity of Liouville theorems. Therefore in Section 3 we also briefly discuss Liouville-type results for the Lotka-Volterra system (1) with coefficients b_1, b_2 not necessarily positive and $d_1 \neq d_2$, see Theorem 7 and Remark 8.

3 Liouville theorems

We will use the following modification of [6, Lemma 2.1].

Proposition 4. Assume (11). Let $h \in C([0, \infty))$, $h(s) > 0$ for $s > 0$. Let $w \in C^{2,1}(X \times \mathbb{R}) \cap C(\overline{X} \times \mathbb{R})$ be bounded and satisfy the inequality

$$(w_t - \Delta w) \operatorname{sign}(w) \leq -h(|w|) \quad \text{in } X \times \mathbb{R}, \quad (16)$$

and the boundary condition $w = 0$ on $\partial \mathbb{R}_+^n \times \mathbb{R}$ if $X = \mathbb{R}_+^n$. Then $w \equiv 0$.

Proof. Assume on the contrary $w \not\equiv 0$. Since $-w$ also satisfies the assumptions of Proposition 4, we may assume

$$W := \sup_{X \times \mathbb{R}} w > 0.$$

Fix $(x^*, t^*) \in X \times \mathbb{R}$ such that $w(x^*, t^*) \geq W/2$. For each $\varepsilon > 0$ set

$$w_\varepsilon(x, t) := w(x, t) - \varepsilon|x - x^*|^2 - \varepsilon(\sqrt{(t - t^*)^2 + 1} - 1), \quad x \in X, \quad t \in \mathbb{R}.$$

Since $w_\varepsilon(x^*, t^*) = w(x^*, t^*) > 0$, $w_\varepsilon(x, t) \rightarrow -\infty$ as $|x| + |t| \rightarrow \infty$, $w_\varepsilon(x, t) < 0$ if $X = \mathbb{R}_+^n$ and $x \in \partial \mathbb{R}_+^n$, there exists $(x_\varepsilon, t_\varepsilon) \in X \times \mathbb{R}$ satisfying $w_\varepsilon(x_\varepsilon, t_\varepsilon) = \sup_{X \times \mathbb{R}} w_\varepsilon$. Notice that

$$W \geq w(x_\varepsilon, t_\varepsilon) \geq w_\varepsilon(x_\varepsilon, t_\varepsilon) \geq w_\varepsilon(x^*, t^*) = w(x^*, t^*) \geq \frac{W}{2} > 0,$$

and

$$(w_\varepsilon)_t(x_\varepsilon, t_\varepsilon) = 0, \quad \Delta w_\varepsilon(x_\varepsilon, t_\varepsilon) \leq 0.$$

Consequently,

$$\begin{aligned} 0 &\leq (w_\varepsilon)_t(x_\varepsilon, t_\varepsilon) - \Delta w_\varepsilon(x_\varepsilon, t_\varepsilon) \\ &= w_t(x_\varepsilon, t_\varepsilon) - \Delta w(x_\varepsilon, t_\varepsilon) - \varepsilon \frac{t_\varepsilon - t^*}{\sqrt{(t_\varepsilon - t^*)^2 + 1}} + 2\varepsilon n \\ &\leq -h(w(x_\varepsilon, t_\varepsilon)) + \varepsilon + 2\varepsilon n \\ &\leq -\inf_{W \geq s \geq W/2} h(s) + \varepsilon + 2\varepsilon n. \end{aligned}$$

Since the first term on the right hand side is negative and independent of ε , we arrive at a contradiction if ε is small enough. \square

Now we are ready to prove Liouville-type theorems for system (12).

Proof of Theorem 3. If $X = \mathbb{R}_+^n$ and (u, v) is a nonnegative solution of the Neumann problem then extending (u, v) by

$$\left. \begin{aligned} u((-x_1, x_2, \dots, x_n), t) &:= u((x_1, x_2, \dots, x_n), t), \\ v((-x_1, x_2, \dots, x_n), t) &:= v((x_1, x_2, \dots, x_n), t), \end{aligned} \right\} \quad x \in \mathbb{R}_+^n, \quad t \in \mathbb{R},$$

we obtain a nonnegative solution of (12) with $X = \mathbb{R}^n$. Consequently, it is sufficient to consider the Dirichlet problem and the case $X = \mathbb{R}^n$.

If $X = \mathbb{R}^n$ and (u, v) is a nonnegative solution of (12) which is not identically zero then by doubling and scaling arguments we may assume that (u, v) is bounded. In fact, assume

that $(u+v)(x_k, t_k) \rightarrow \infty$ for some $(x_k, t_k) \in \mathbb{R}^n \times \mathbb{R}$. Set $M := (u+v)^{(q+r-1)/2}$. Then the Doubling lemma [14, Lemma 5.1] guarantees the existence of $(\tilde{x}_k, \tilde{t}_k)$ such that $M(\tilde{x}_k, \tilde{t}_k) \geq M(x_k, t_k) \rightarrow \infty$ and $M(x, t) \leq 2M(\tilde{x}_k, \tilde{t}_k)$ for all (x, t) satisfying $|x - \tilde{x}_k| + (t - \tilde{t}_k)^{1/2} \leq k\lambda_k$, where $\lambda_k := 1/M(\tilde{x}_k, \tilde{t}_k)$. It is easily seen that the rescaled functions

$$\begin{aligned}\tilde{u}(y, s) &:= \lambda_k^{2/(q+r-1)} u(\tilde{x}_k + \lambda_k y, \tilde{t}_k + \lambda_k^2 s), \\ \tilde{v}(y, s) &:= \lambda_k^{2/(q+r-1)} v(\tilde{x}_k + \lambda_k y, \tilde{t}_k + \lambda_k^2 s)\end{aligned}$$

converge locally uniformly to a nonnegative nontrivial bounded solution of (12).

Hence, we may assume that (u, v) is a nonnegative bounded solution of (12) with $X = \mathbb{R}^n$ or a nonnegative bounded solution of the Dirichlet problem. Now [12, Lemma 7.1(i)] guarantees the existence of $K, C > 0$ such that the function $w := u - Kv$ satisfies

$$(w_t - \Delta w) \operatorname{sign}(w) \leq -C(u + Kv)^{q+r-1} |w| \leq -C|w|^{q+r},$$

hence Proposition 4 yields $u = Kv$. Our assumption $c_1 c_2 > b_1 b_2$ guarantees that u solves the scalar equation

$$u_t - \Delta u = cu^{r+q}$$

with some $c > 0$ (and satisfies the Dirichlet boundary condition if $X = \mathbb{R}_+^n$). Consequently, it is sufficient to use the Liouville theorems in [3, 15, 17]. \square

Remark 5. Assume $r = 1$. Then the constant K in the proof of Theorem 3 can be computed explicitly: $K = [(c_1 + b_2)/(c_2 + b_1)]^{1/q}$ (see [12]). Notice also that if $r = q = 1$ and $w = u - Kv$ then $w_t - \Delta w = -(b_1 u + b_2 v)w$.

In the proof of the existence of periodic solutions we will also need estimates based on the following Liouville theorem.

Theorem 6. Assume (11) and $n \leq 5$. Let b_1, b_2, c_1, c_2 be constants satisfying (3), $\lambda \in [0, 1]$ and $K := (c_1 + b_2)/(c_2 + b_1)$. Let (u, v) be a nonnegative solution of the system

$$\left. \begin{aligned} u_t - \Delta u &= \lambda u(-b_1 u + c_1 v) + (1 - \lambda)K^3 v^2 \\ v_t - \Delta v &= \lambda v(-b_2 v + c_2 u) + (1 - \lambda)u^2 \end{aligned} \right\} \quad \text{in } X \times \mathbb{R}, \quad (17)$$

complemented by the homogeneous Dirichlet or Neumann boundary conditions if $X = \mathbb{R}_+^n$. In the case of the Dirichlet boundary condition assume also that (u, v) is bounded. Then $u \equiv v \equiv 0$.

Proof. The proof is almost the same as the proof of Theorem 3 with $q = r = 1$. Due to Remark 5, the function $w := u - Kv$ satisfies

$$\begin{aligned} w_t - \Delta w &= -\lambda(b_1 u + b_2 v)w + (1 - \lambda)K(K^2 v^2 - u^2) \\ &= -(\lambda(b_1 u + b_2 v) + (1 - \lambda)K(u + Kv))w \end{aligned}$$

hence

$$(w_t - \Delta w) \operatorname{sign}(w) \leq -\tilde{C}|w|^2,$$

and $u = Kv$ due to Proposition 4. Consequently, u solves the scalar equation

$$u_t - \Delta u = (\lambda c + (1 - \lambda)K)u^2,$$

where $c = c_1/K - b_1 > 0$. The scalar Liouville theorems in [3, 15] guarantee $u \equiv 0$. \square

In the rest of this section we consider scaling invariant problems corresponding to the Lotka-Volterra systems without assumption (3) or in the case of unequal diffusion coefficients $d_1 \neq d_2$.

Theorem 7. *Assume $b_1 = b_2 = 0$, $c_1, c_2 > 0$ and $n \leq 5$. Let (u, v) be a nonnegative bounded solution of (10) with $X = \mathbb{R}^n$. Then either $(u, v) \equiv (C, 0)$ or $(u, v) \equiv (0, C)$ for some $C \geq 0$.*

Proof. Scaling arguments show that we may assume $c_1 = c_2 = 1$. The function $w := u - v$ is a bounded entire solution of the linear heat equation hence $w \equiv D$ for some constant D (see [4, Theorem 1]). W.l.o.g. we may assume $D \geq 0$. If $D = 0$ then $u = v$ and the Liouville theorem [18, Theorem 21.2] guarantees $u \equiv v \equiv 0$. If $D > 0$ then given $p \in (1, \min(2, 1 + 2/n)]$ there exists $d = d(p, D) > 0$ such that

$$v_t - \Delta v = uv = (v + D)v \geq dv^p,$$

and the Fujita theorem [18, Theorem 18.1] together with the comparison principle imply $v \equiv 0$. \square

The existence of semi-trivial entire solutions of the form $(C, 0)$ and $(0, C)$ with $C > 0$ disables one to use standard scaling arguments to prove a priori estimates of solutions in a straightforward way. However, at least in the case of similar elliptic systems, existence of semi-trivial entire solutions represents just a technical difficulty and the scaling arguments do apply, see [21].

Remark 8. If one considers (10) with $X = \mathbb{R}^n$, $b_1, b_2 \leq 0$ and $c_1, c_2 > 0$, for example, then the function $w := \sqrt{uv}$ satisfies $w_t - \Delta w \geq cw^2$ for some $c > 0$, hence the Fujita theorem [18, Theorem 18.1] guarantees $w \equiv 0$ if $n \leq 2$ (and similar result can be obtained for the Dirichlet problem in the halfspace if $n = 1$, see [18, Remark 18.6(i)]). If $b_1, b_2 \neq 0$ then this implies $u \equiv v \equiv 0$, and these Liouville theorems for (10) enable one to prove universal estimates of solutions of (1) with $d_1 = d_2 = 1$. In addition, the Fujita-type theorems mentioned above and comparison with suitable subsolutions enables one to prove the Liouville theorems even for the generalization of (10) with unequal diffusion coefficients $d_1 \neq d_2$, see [8]. However, the restrictions $n \leq 2$ and $n = 1$ (in the case of the Dirichlet problem) seem to be far from optimal.

4 Universal estimates

Proof of Theorem 1. The proof follows those of [15, Theorem 3.1 and 4.1] and we just sketch it.

In order to prove estimate (8) with $C_1 = C_2 = 1$ we will follow the proof of [15, Theorem 3.1(i)]. Assume that estimate (8) fails. Then, for $k = 1, 2, \dots$, there exist nonempty open sets Ω_k , $T_k \in (0, \infty]$, coefficients $a_{i,k}, b_{i,k}, c_{i,k}$, $i = 1, 2$, satisfying (6) with Ω replaced by Ω_k , solutions (u_k, v_k) of (7) with $\Omega, T, a_1, a_2, b_1, b_2, c_1, c_2$ replaced by $\Omega_k, T_k, a_{1,k}, a_{2,k}, b_{1,k}, b_{2,k}, c_{1,k}, c_{2,k}$ and points $(y_k, \tau_k) \in D_k := \Omega_k \times (0, T_k)$ such that

$$M_k(y_k, \tau_k) > 2k(1 + d_P^{-1}((y_k, \tau_k), \partial D_k)), \quad (18)$$

where $M_k := \sqrt{u_k + v_k}$ and

$$d_P((x, t), (y, \tau)) := |x - y| + |t - \tau|^{1/2}$$

denotes the parabolic distance. The Doubling lemma [14, Lemma 5.1] guarantees the existence of $(x_k, t_k) \in D_k$ such that

$$\begin{aligned} M_k(x_k, t_k) &\geq M_k(y_k, \tau_k), & M_k(x_k, t_k) &> 2kd_P^{-1}((x_k, t_k), \partial D_k), \\ M_k(x, t) &\leq M_k(x_k, t_k) & \text{whenever } d_P((x, t), (x_k, t_k)) &\leq k\lambda_k, \end{aligned}$$

where

$$\lambda_k := M_k^{-1}(x_k, t_k) \rightarrow 0.$$

Set

$$\begin{aligned} \tilde{u}_k(y, s) &:= \lambda_k^2 u_k(x_k + \lambda_k y, t_k + \lambda_k^2 s), \\ \tilde{v}_k(y, s) &:= \lambda_k^2 v_k(x_k + \lambda_k y, t_k + \lambda_k^2 s), \\ \tilde{a}_{1,k}(y, s) &:= a_{1,k}(x_k + \lambda_k y, t_k + \lambda_k^2 s), \end{aligned}$$

and define $\tilde{a}_{2,k}, \tilde{b}_{1,k}, \tilde{b}_{2,k}, \tilde{c}_{1,k}, \tilde{c}_{2,k}$ analogously. Then $(\tilde{u}_k, \tilde{v}_k)$ solve the system

$$\begin{aligned} \tilde{u}_t - d_1 \Delta \tilde{u} &= \tilde{u}(\tilde{a}_{1,k} \lambda_k^2 - \tilde{b}_{1,k} \tilde{u} + \tilde{c}_{1,k} \tilde{v}), \\ \tilde{v}_t - d_2 \Delta \tilde{v} &= \tilde{v}(\tilde{a}_{2,k} \lambda_k^2 - \tilde{b}_{2,k} \tilde{v} + \tilde{c}_{2,k} \tilde{u}), \end{aligned}$$

in the corresponding rescaled region, $\tilde{u}_k(0, 0) + \tilde{v}_k(0, 0) = 1$ and $u_k + v_k \leq 4$ in $\tilde{D}_k := \{y \in \mathbb{R}^n : |y| < k/2\} \times (-k^2/4, k^2/4)$. Passing to a subsequence we may assume $\tilde{b}_{i,k}(0, 0) \rightarrow \tilde{b}_i > 0$ and $\tilde{c}_{i,k}(0, 0) \rightarrow \tilde{c}_i > 0$, $i = 1, 2$, where $\tilde{c}_1 \tilde{c}_2 > \tilde{b}_1 \tilde{b}_2$. Now standard regularity estimates guarantee that a subsequence of $(\tilde{u}_k, \tilde{v}_k)$ converges to a nontrivial nonnegative solution of (10) with $X = \mathbb{R}^n$ and b_1, b_2, c_1, c_2 replaced by $\tilde{b}_1, \tilde{b}_2, \tilde{c}_1, \tilde{c}_2$, which contradicts Theorem 3.

If Ω is smooth and the solution (u, v) satisfies the homogeneous Dirichlet or Neumann boundary condition on $\partial\Omega \times (0, T)$ then one just has to modify the proof in the same way as in the proof of [15, Theorem 4.1]. If $a_1 = a_2 = 0$ and b_1, b_2, c_1, c_2 are constants then it is sufficient to replace the inequality (18) with

$$M_k(y_k, \tau_k) > 2kd_P^{-1}((y_k, \tau_k), \partial D_k)$$

and notice that λ_k need not converge to zero (cf. the proof of [15, Theorem 3.1(ii)]). \square

In the same way as in the proof Theorem 1, using Theorem 6 instead of Theorem 3 and assuming (9), we obtain the following universal bounds for periodic solutions of the homotopy problem

$$\left. \begin{aligned} u_t - \Delta u &= \lambda u(a_1 - b_1 u + c_1 v) + (1 - \lambda)(\Lambda u + K^3 v^2) \\ v_t - \Delta v &= \lambda v(a_2 - b_2 v + c_2 u) + (1 - \lambda)(\Lambda v + u^2) \\ u = v &= 0 \quad \text{on } \partial\Omega \times (0, \infty), \end{aligned} \right\} \quad \text{in } \Omega \times (0, \infty), \quad (19)$$

where

$$\lambda \in [0, 1], \quad \Lambda > 0, \quad K = K(x, t) := \frac{c_1 + b_2}{c_2 + b_1}. \quad (20)$$

Theorem 9. *Assume (9) and (20). Then there exists a positive constant C depending only on $\Omega, T, \Lambda, a_1, a_2, b_1, b_2, c_1, c_2$ such that any positive T -periodic solution (u, v) of (19) satisfies*

$$u(x, t) + v(x, t) \leq C \quad \text{for all } (x, t) \in \Omega \times (0, \infty). \quad (21)$$

5 Periodic solutions

In this section we assume (9). In addition, by $\mathcal{X} := BUC(\Omega \times (0, T))$ we denote the space of bounded uniformly continuous functions equipped with the L^∞ -norm $\|\cdot\|_\infty$, and $w^+(x, t) := \max(w(x, t), 0)$. Without fearing confusion, by $\|\cdot\|_\infty$ we denote both the norm in $L^\infty(\Omega \times (0, T))$ and $L^\infty(\Omega)$.

In the proof of our main result we will need the following proposition on (possibly sign-changing) solutions.

Proposition 10. *Let $\Omega \subset \mathbb{R}^n$ be a C^3 -smooth bounded domain, $T > 0$ and $f \in \mathcal{X}$. Then the scalar periodic problem*

$$\left. \begin{aligned} w_t - \Delta w &= f && \text{in } \Omega \times (0, T), \\ w &= 0 && \text{on } \partial\Omega \times (0, T), \\ w(\cdot, 0) &= w(\cdot, T) && \text{in } \Omega, \end{aligned} \right\} \quad (22)$$

possesses a unique solution w . In addition, the mapping $\mathcal{K} : \mathcal{X} \rightarrow \mathcal{X} : f \mapsto w^+$ is compact.

Proof. The assertion was proved in [1] for f being Hölder continuous. In our case it is sufficient to combine this result with standard mollifying arguments and L^∞ - and smoothing estimates for the corresponding initial value problem. For example, if w is a periodic solution of (22) then the variation-of-constants formula yields the estimate

$$\|w(\cdot, t_2)\|_\infty \leq e^{-\Lambda_1(t_2 - t_1)} \|w(\cdot, t_1)\|_\infty + (t_2 - t_1) \|f\|_\infty,$$

and, consequently, the periodicity of w guarantees $\|w\|_\infty \leq C(T) \|f\|_\infty$. \square

We will also need the following proposition on the adjoint eigenvalue problem.

Proposition 11. *Let $\Omega \subset \mathbb{R}^n$ be a C^3 -smooth bounded domain and $T > 0$. Then there exists $\Lambda_1^T > 0$ such that the problem*

$$\left. \begin{aligned} -\varphi_t - \Delta\varphi &= \Lambda_1^T \varphi && \text{in } \Omega \times (0, T), \\ \varphi &= 0 && \text{on } \partial\Omega \times (0, T), \\ \varphi(\cdot, 0) &= \varphi(\cdot, T) && \text{in } \Omega, \end{aligned} \right\} \quad (23)$$

possesses a positive solution φ .

Proof. The result follows again from [1]; cf. also [5, (18)]. \square

Proof of Theorem 2. Let \mathcal{K} be the compact mapping from Proposition 10. First notice that fixed points of the compact operator

$$\begin{aligned} \mathcal{T} : \mathcal{X} \times \mathcal{X} &\rightarrow \mathcal{X} \times \mathcal{X} \\ \mathcal{T}(u, v) &:= (\mathcal{K}(u(a_1 - b_1 u + c_1 v)), \mathcal{K}(v(a_2 - b_2 v + c_2 u))), \end{aligned}$$

correspond to nonnegative periodic solutions of our problem. In fact, if $(u, v) \neq (0, 0)$ is a fixed point of \mathcal{T} then $u = w^+$ and $v = z^+$, where (w, z) are T -periodic solutions of

$$\left. \begin{aligned} w_t - \Delta w &= w^+(a_1 - b_1 w^+ + c_1 z^+) \\ z_t - \Delta z &= z^+(a_2 - b_2 z^+ + c_2 w^+) \end{aligned} \right\} \quad \begin{aligned} &\text{in } \Omega \times (0, \infty), \\ &\text{on } \partial\Omega \times (0, \infty), \end{aligned}$$

$$w = z = 0$$

and the maximum principle guarantees $w, z \geq 0$.

Notice also that nontrivial nonnegative periodic solutions of $(1)_D$ are positive. In fact, by the maximum principle it is sufficient to exclude the possibility $v \equiv 0$ (or $u \equiv 0$). If, for example, $v \equiv 0$ then the assumptions $a_1 < \Lambda_1$ and $b_1 > 0$ guarantee that the function $t \mapsto \int_{\Omega} u(\cdot, t) \varphi_1 dx$ (where φ_1 is a positive eigenfunction of the negative Dirichlet Laplacian corresponding to the eigenvalue Λ_1) is time-decreasing which contradicts the periodicity of u . In fact, multiplying the first equation in (7) by φ_1 and integrating over Ω yields

$$\frac{d}{dt} \int_{\Omega} u(\cdot, t) \varphi_1 dx = \int_{\Omega} (a_1 - \Lambda_1) u(\cdot, t) \varphi_1 dx - \int_{\Omega} b_1 u^2(\cdot, t) \varphi_1 dx < 0.$$

We will prove the existence of a nontrivial fixed point of \mathcal{T} (hence a positive periodic solution of $(1)_D$) by computing the Leray-Schauder degree $d(r) := \deg(I - \mathcal{T}, B_r, 0)$ for small and large r , where I denotes the identity and B_r is the ball in $\mathcal{X} \times \mathcal{X}$ with radius r centered at zero. In fact, we will prove $d(r) = 1$ if $r > 0$ is sufficiently small and $d(r) = 0$ if $r > 0$ is large enough.

First consider r small. The assertion $d(r) = 1$ follows by using the homotopy

$$\mathcal{T}_{\lambda}(u, v) := (\mathcal{K}(\lambda u(a_1 - b_1 u + c_1 v)), \mathcal{K}(\lambda v(a_2 - b_2 v + c_2 u))), \quad \lambda \in [0, 1].$$

To show that this homotopy is admissible, assume that there exists a nontrivial fixed point (u, v) of \mathcal{T}_{λ} satisfying $\|(u, v)\|_{\infty} = r \ll 1$. Fix t such that $\|(u(\cdot, t), v(\cdot, t))\|_{\infty} = r$. W.l.o.g.

we may assume $\|u(\cdot, t)\|_\infty = r$. Notice that (u, v) is a positive periodic solution of $(1)_D$ with the right-hand sides multiplied by λ , and the variation-of-constants formula yields

$$\begin{aligned} r &= \|u(\cdot, t+T)\|_\infty \leq e^{-(\Lambda_1 - \lambda \max a_1)T} \|u(\cdot, t)\|_\infty + CT\lambda(\|u\|_\infty^2 + \|u\|_\infty \|v\|_\infty) \\ &\leq e^{-(\Lambda_1 - \max a_1)T} r + 2CTr^2, \end{aligned}$$

which yields a contradiction for r small.

Now consider r large. We will use the homotopy

$$\begin{aligned} \mathcal{T}_\lambda(u, v) &:= (\mathcal{K}(\lambda u(a_1 - b_1 u + c_1 v) + (1 - \lambda)(\Lambda u + K^3 v^2)), \\ &\quad \mathcal{K}(\lambda v(a_2 - b_2 v + c_2 u) + (1 - \lambda)(\Lambda v + u^2))), \quad \lambda \in [0, 1], \end{aligned}$$

where $\Lambda := \Lambda_1^T + 1$. Estimates in Theorem 9 guarantee that this homotopy is admissible if r is large enough. Hence it is sufficient to show that problem (19) does not possess positive periodic solutions if $\lambda = 0$.

Assume on the contrary that (u, v) is a positive T -periodic solution of the system

$$\begin{cases} u_t - \Delta u = \Lambda u + K^3 v^2 \\ v_t - \Delta v = \Lambda v + u^2 \\ u = v = 0 \end{cases} \quad \begin{cases} \text{in } \Omega \times (0, \infty), \\ \text{on } \partial\Omega \times (0, \infty). \end{cases}$$

Multiplying the first equation by the eigenfunction φ from Proposition 11, integrating over $\Omega \times (0, T)$ and using integration by parts we obtain

$$\Lambda_1^T \int_0^T \int_\Omega u \varphi \, dx \, dt \geq \Lambda \int_0^T \int_\Omega u \varphi \, dx \, dt,$$

which yields a contradiction. This concludes the proof. \square

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